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Transition on the relationship between fractal dimension and Hurst exponent in the long-range connective sandpile models

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ABSTRACT

The relationships between the Hurst exponent H and the power-law scaling exponent B in a new modification of sandpile models, i.e. the long-range connective sandpile (LRCS) models, exhibit a strong dependence upon the system size L. As L decreases, the LRCS model can demonstrate a transition from the negative to positive correlations between H- and B-values. While the negative and null correlations are associated with the fractional Gaussian noise and generalized Cauchy processes, respectively, the regime with the positive correlation between the Hurst and power-law scaling exponents may suggest an unknown, interesting class of the stochastic processes.

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Sandpile dynamics and self-organized criticality (SOC) are known to be exhibited in many natural and social phenomena including earthquakes, forest fires, rainfalls, landscapes, drainage networks, stock prices, traffic jams, and so on. Since the original nearest-neighboring sandpile model was introduced by Bak et al. [1,2], various numerical and analytical studies of modified sandpile models have been a considerable subject of researches, e.g. [3–11]. Among them, the *annealed* random-neighbor sandpile models where an avalanche can propagate within the system were first (perhaps) proposed by Christensen and Olami [4] and then extensively studied on a long-range connected (small-world) network by, for example, de Arcangelis and Herrmann [7], Lahtinen et al. [9], and Chen et al. [10,11].

We have previously proposed a *long-range connective sandpile* (LRCS) model by introducing randomly remote connections between two separated cells [10–13]. For a square lattice of L by L cells, we randomly throw sands, one at a time, onto the grid. In the original Bak–Tang–Wiesenfeld (BTW) sandpile model, once the total amount of the accumulated sands on a single cell reaches the threshold amount of four, they will be redistributed to the four adjacent cells (the nearest neighbors) or lost off the edge of the grid. Our modified LRCS model differs from the BTW model in view of releasing toppled grains to four nearest neighboring cells. The modified rule of randomly internal connections is very similar to the implementation of Watts and Strogatz [14]. For any particular cell, when the accumulated grains exceed the threshold and redistribution occurs, one of the original nearest neighbor connections confronts a chance with the *long-range connective probability* P_c of redirecting to a randomly chosen, distant cell and so the original connection is replaced by a randomly chosen mesh that may be far from the toppling cell. For a scheme of the distribution process of the LRCS model please refer to Fig. 1. We have furthermore assumed that P_c depends strongly on topographic change induced by the last event [11–13]. Consider that topographic height of the sandpile $\mathbf{x}$ at the iteration step t is $\boldsymbol{Z}_t(\mathbf{x})$. At the next iteration step $t+1$, due to the throw of single grain on the grid, it changes from $\boldsymbol{Z}_t(\mathbf{x})$ to $\boldsymbol{Z}_{t+1}(\mathbf{x})$. Therefore, total change in the topographic height of the sandpile is $\Delta Z(t+1) = \sum_{x_l} |Z_{t+1}(x_l) - Z_t(x_l)|$. We then define $P_c(t+1) = [\Delta Z(t+1)/\alpha L^2]^3$. The meaning for the coefficient α is basically like the normalization constant, which makes the value of the connective probability P_c range between 0 and 1. The simulation throughout this study was performed in the "stop-and-go" mode. The LRCS model after a large avalanche can thus evoke a high value of connective probability P_c, motivated by that a more active earthquake fault system will have higher probability to establish long-range connection due to the fault activity, the change in pore fluid pressure or the dynamic triggering of seismic waves. By using such self-adapted probability threshold P_c of remote connection, the self-adapted LRCS model demonstrates a state of *intermittent criticality* [15–17], in which the

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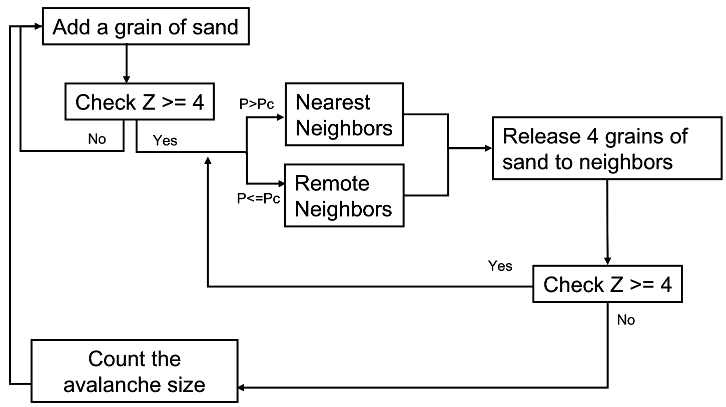

Fig. 1. Flowchart of the LRCS model illustrating the criterion about random distribution to remote site of toppling sand.

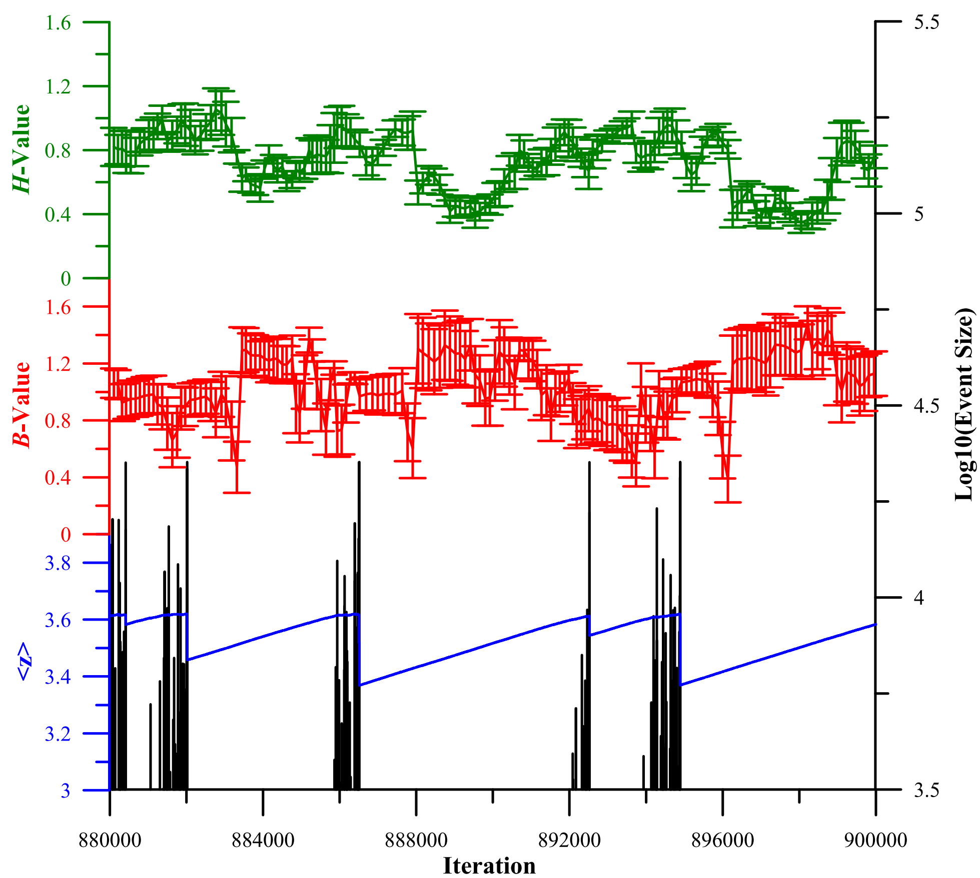

Fig. 2. Simulation for a square lattice of 150 by 150 cells. Blue line represents the dynamic variable $\langle Z\rangle(t)$ of the average topographic height of the LRCS model. Green and red lines are the Hurst exponent H of avalanche sizes and the power-law exponent B of frequency-size distribution, respectively. Error bars show the 95% confidence intervals. Also shown are the time occurrences of avalanches with sizes > 3162 (black bars). (For interpretation of colors in this figure, the reader is referred to the web version of this Letter.)

sandpile quasi-periodically approaches and retreats from the critical state.

In the LRCS model with self-adapted P_c, the dynamic variable of the spatially averaged amount of grains on board $\langle Z\rangle(t)$ $(=(\sum_{i=1}^{L^2} Z_i(t))/L^2$, blue lines in Figs. 2 and 3) is often punctuated towards smaller values by large events (black bars in Figs. 2 and 3). The large fluctuation in $\langle Z\rangle(t)$ is an important feature mimicking the intermittent criticality [15–20]. Large avalanches are then followed by a period of quiescence and a new approach back toward the critical state (Fig. 2) [11–13]. Such process is similar to the dynamical process of the earthquake fault system which repeats by reloading energy and rebuilding correlation lengths towards criticality and the next great event [18–20,33,34]. For more details about the LRCS model, we refer the readers to our previous papers [11–13].

In this Letter we investigate the temporal variations in the power-law exponent B of the frequency-size distributions and in the Hurst exponent H of avalanche sizes for various system sizes L of the LRCS models. To trace variations in B and H with respect to time the sliding window technique was used. We selected 500 events for every time window to calculate Bs and Hs, and then shifted 50 events to calculate the successive B- and H-values of the next window. For the calculation of B, we applied the data binning technique proposed by Christensen and Moloney [21] to

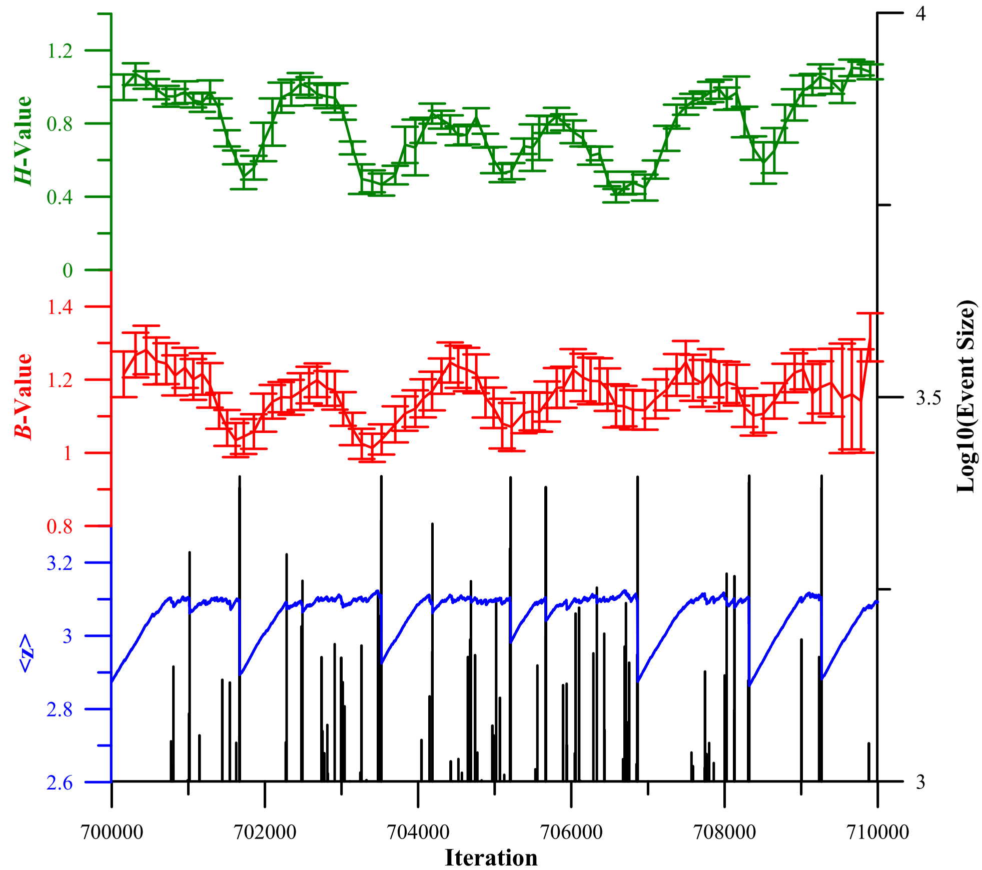

Fig. 3. Simulation for a square lattice of 50 by 50 cells. Blue line represents the dynamic variable $\langle Z\rangle(t)$ of the average topographic height of the LRCS model. Green and red lines are the Hurst exponent H of avalanche sizes and the power-law exponent B of frequency-size distribution, respectively. Error bars show the 95% confidence intervals. Also shown are the time occurrences of avalanches with sizes > 1000 (black bars). (For interpretation of colors in this figure, the reader is referred to the web version of this Letter.)

reduce the noise effect of large avalanches, i.e. the effect of finite statistics, and then adopted the weighted least-square regression to fit the frequency-size distribution. As for the calculation of H, a brief summary of the R/S analysis is given below. The R/S analysis utilizes two factors: one is the range R, which is the difference between maximum and minimum amounts of accumulated departure of time series from the mean over a time span τ, and the other one the standard deviation S over that time span. The rescaled range is defined as the ratio of R and S, i.e. R/S. Analyzing a variety of time series of natural phenomena, the avalanche size for example, it has been concluded that the ratio R/S is very well described by the empirical relation $(R/S)(\tau) = (\tau/2)^H$, where H is the Hurst exponent. For the independent random process, with no correlations among samples, $H = 0.5$. The observational time series is persistent for $H > 0.5$ whereas the sequence shows the anti-persistent behavior for $H < 0.5$. The concepts of persistent and anti-persistent memories in time are well defined for the non-linear processes [22].

Two examples of the LRCS models with the grid sizes of 150 by 150 and 50 by 50 are given in Figs. 2 and 3, respectively, demonstrating the temporal variations in the B (red lines) and H (green lines) values. Error bars show the 95% confidence intervals. We had previously found that, for large grid sizes of sandpiles ($L > 100$), the change in H has fluctuation in the opposite sense to the variation in B (Fig. 2) [13]. While the B-value usually increases following a large avalanche, the H-value usually decreases. Larger B means more small events whose superposition results in a higher-frequency noise in terms of the sandpile topography, hence smaller H (personal communication with the reviewer). The H-value then increases prior to the next large avalanche and the B-value decreases. Comparing B with H in Fig. 2, we could find a strikingly negative correlation between two exponents of B and H. Here in this Letter, interestingly, the negative correlation is transferred into a positive correlation (Fig. 3) when simulations are performed over smaller grid sizes of sandpile systems ($L < 100$). Systematic simulations over various dimensions L of sandpile models from 30 to 200 had been conducted and the result on the relationships between B and H is shown in Fig. 4. The scaling exponent B was initially calculated with all binning points from the frequency-size distribution (red line in Fig. 4). Taking the potential finite-size effect into account, we also calculated the B-value from the frequency-size distribution without some tail binning points for those simulations on smaller grid sizes of sandpiles. We only fitted the frequency-size distribution within the size range between 1 and $\sim$1000 for those small L sandpiles. This turns out to display a clearer positive correlation between B and H (blue line in Fig. 4). Another widely used correction on the finite-size effect evokes the addition of a correction factor in a form like $\exp(-s/s_c)$ (please refer to Ref. [21]) as fitting the frequency-size distribution. We have furthermore fitted our data with that kind of correction factors in three sandpiles with $L = 50, 100$ and 150, respectively. The black stars in Fig. 4 indicate that the early strategy for eliminating the finite-size effect works quite well and confirm the observed transition in the B–H correlation.

Since the system-size dependence could be revealed by the moment analysis of sanpile models we furthermore conduct the first moment calculation of avalanche sizes for various dimensions L of the LRCS models. Let $P(s)$ denote the distribution function of

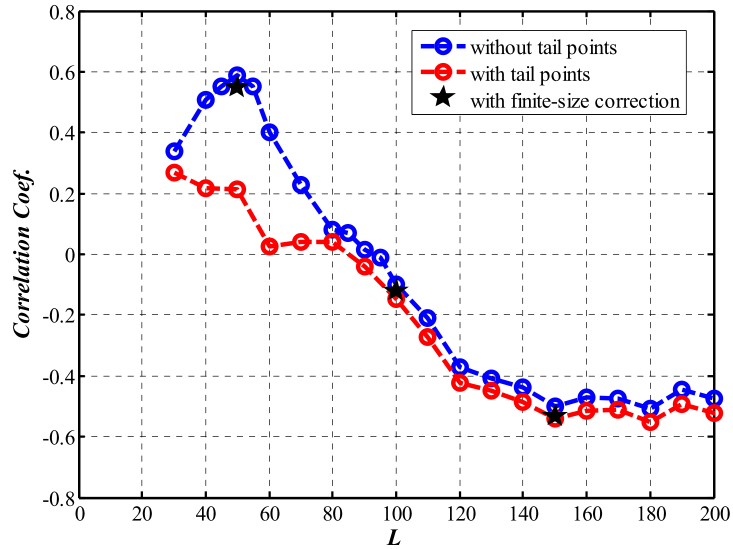

Fig. 4. Correlation coefficients between B and H as a function of system sizes L for the LRCS models. One can clearly see a transition from positive correlations to negative correlations as L increases. For the explanation please refer to the text. (For interpretation of colors in this figure, the reader is referred to the web version of this Letter.)

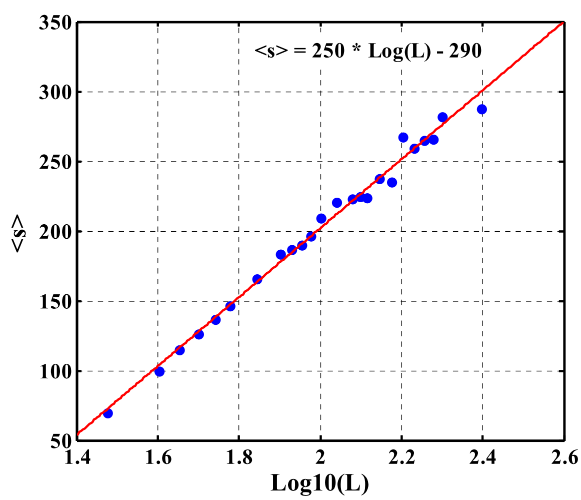

Fig. 5. Scaling relation between the first moment $\langle s \rangle$ of avalanche size distribution and the system size L for the LRCS models.

avalanche sizes s. We define the first moment of avalanche size distribution on a lattice of size L as $\langle s \rangle_L = \int sP(s)\,ds$. In Fig. 5 we show the results obtained from the first moment calculation of the distribution $P(s)$ for the LRCS models. The LRCS models exhibit a striking scaling with the linear system size L of the first moment of avalanche sizes as $\langle s \rangle \propto \log(L)$. Such a scaling is consistent with the scaling relation obtained by Majumdar and Dhar (Eq. (5.9) in [23]) if the average avalanche size $\langle s \rangle$ is proportional to the average number of successful remote rewirings in the LRCS model.

The important implication with the senses of earthquake statistics and stochastic processes could be drawn from the present study. To seismologists the negative correlation between two scaling exponents of B and H is fundamentally important for understanding earthquake statistics and rupturing processes, e.g. [24,25], and has also been suggested in other conceptual models of earthquake fault systems, e.g. [26]. For instance, a self-affine asperity model proposed by Hallgass et al. [26] exhibits the dependence of the scaling exponent in the frequency-size distribution, i.e. the Gutenberg–Richter law, on the roughness of fault geometry which is controlled by the Hurst exponent in the fractional Brownian fault profiles. In their numerical simulations (Fig. 4 in [26]) the observed negative correlation between those two scaling exponents is attributed to the self-affine nature of the considered fault ensembles. It is worth to mention that, based on the self-affine traces of fractional Brownian motion, Voss [27] has presented a relation of fractal dimension D and Hurst exponent H, i.e. $D = 2 - H$. Notice that the scaling exponent B of the frequency-size distribution shares the geometric meaning of fractal dimension [28,29]. The sandpile models have since its invention represented a conceptual paradigm of self-organized earthquake fault systems [30]. Therefore, while we cannot find such a negative correlation in the original BTW sandpile models [13], our LRCS models in the sense of the negative correlation between B and H seem consistent with past studies of earthquake fault systems [25,26]. The challenge of our present study is then to understand how a negative correlation between B and H is transferred into a positive correlation as the system size L decreases (Fig. 4). Investigating this transition is sort of beyond the scope of the present Letter. We will postpone this to a future work.

Stochastic processes are characterized by their auto-correlation function. Mathematically speaking, while the negative correlation between the fractal dimension and the Hurst exponent could be derived in the process of fractional Brownian motion (fBm)/fractional Gaussian noise (fGn), e.g. [27], the independence upon B of H was recently established in the generalized Cauchy (gC) process [31,32]. The fGn (the derivative of fBm) is a traditional stationary, self-affine stochastic process with the correlation function $C(\tau) = \frac{1}{2}(|\tau + 1|^{2H} - 2|\tau|^{2H} + |\tau - 1|^{2H})$, where H is a self-affinity index (the Hurst exponent) and τ is the lag. On the hand, the gC process is given by a correlation function with the form $C(\tau) = (1 + |\tau|^{\alpha})^{-\beta/\alpha}$. Here α and β are two parameters related to the fractal dimension and Hurst exponent, respectively, with the relations of $\alpha = 4 - 2D$ and $\beta = 2 - 2H$. The gC class provides flexible power-law correlations and generalizes stochastic models recently discussed in geostatistics, hydrology, network traffics, physics and time series analysis [31,32]. In this context of stochastic processes, the regime we discovered with the positive correlation between the fractal dimension and Hurst exponent may then suggest an unknown, interesting class of the stochastic processes.

To give a final remark, in terminology, the LRCS model proposed here could be called the quarterly self-adaptive random-neighbor BTW model. In terms of *quarterly* that means only one among four nearest-neighbor connections was removed and redirected, and in terms of *self-adaptive* that means the long-range connective probability P_c was recalculated according to the topographic change induced by the last event. Notice that, although other random-neighbor versions of sandpile models have been proposed during last two decades, e.g. [3–9], none of them was self-adaptive in the sense of random connection probability. We have realized [11–13] that self-adapted P_c plays a crucial role to the intermittency in the LRCS model.

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